

Gauge Theories and Geometric Representation Theory

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2023-10-12 @ UNIST (Ulsan National Institute of Science and Technology)

1. Topological Quantum Field Theories (Atiyah - Segal style)

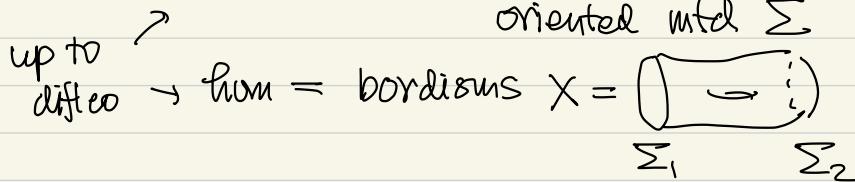
d -dimensional TQFT Σ is a functor (symmetric and monoidal)

$$\text{Bord}_d \xrightarrow{\Sigma} \text{Vect}$$

category of d -dim bordisms

objects = $(d-1)$ -dim. closed

oriented mfld Σ



composition = gluing

disjoint union $\Sigma_1 \sqcup \Sigma_2$

unit object = \emptyset

category of \mathbb{C} -vector spaces

objects = vector spaces

$Z(\Sigma)$

from = linear maps

$Z(X) : Z(\Sigma_1) \rightarrow Z(\Sigma_2)$

composition = usual composition
of linear maps

\otimes tensor product $Z(\Sigma_1) \otimes Z(\Sigma_2)$

unit object = \mathbb{C} $Z(\emptyset) = \mathbb{C}$

Motivation

$\mathcal{F}(X)$ = the space of all fields φ on X

e.g. - maps $X \rightarrow$ target space T

lagrangian
connections
...

$$\mathcal{Z}(X) \stackrel{\text{"def."}}{=} \int_{\mathcal{F}(X)} e^{iL(\varphi)} D\varphi \in \mathbb{C}$$

integral is not mathematically justified

If X has a boundary Σ , we need to fix the boundary value of φ

$$\mathcal{F}(\Sigma) \ni \phi \mapsto \int_{\substack{\varphi \in \mathcal{F}(X) \\ \varphi|_{\Sigma} = \phi}} e^{iL(\varphi)} D(\varphi)$$

This should be a function on $\mathcal{F}(\Sigma)$.

$\mathcal{Z}(\Sigma) \stackrel{\text{"def."}}{=} \text{vector space of functions arising this way.}$

But, path integrals are not mathematically justified. How they fit with TQFT, constructed (at least partially) by Donaldson, Floer, Witten, etc ?

Witten's Chern-Simons theory → different story (omitted today)

Donaldson / Floer theory (4dTQFT)

path integrals are **localized** to integrals over **minimum of L**,

SUSY \rightsquigarrow which are "moduli spaces", finite dimensional manifolds.

$$\circ \Xi(X) = \int_{\text{moduli space}} (\text{differential form}) = \langle [\text{moduli sp.}], \text{cohomology class} \rangle$$

Donaldson invariants are defined in this way.

$$\circ \Xi(\Sigma) = H^*(\text{moduli space on } \Sigma) \quad (\text{This is an approximation. Floer homology is more involved.})$$

TQFT gives a heuristic explanation why moduli spaces and their cohomology appear.

We consider 3d reduction (by S^1) of Donaldson/Floer type theories

defined for (G_c : cpt Lie group
 \mathbf{M} : quaternionic representation
3d N=4 theory)

- Expect:
- X^3 : closed 3-mfd $\rightsquigarrow \Sigma(X) \in \mathbb{C}$
oriented
 - Σ^2 : closed 2-mfd $\rightsquigarrow \Sigma(\Sigma)$: vector space
oriented
 - X^3 with bdry $\partial X = \Sigma^2$ $\rightsquigarrow \Sigma(X) \in \Sigma(\Sigma)$

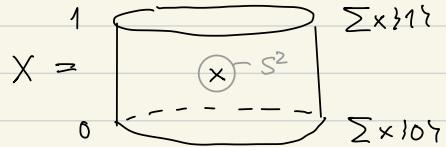
Caveat : We usually do not have a full TQFT. Some of axioms must be dropped.

Therefore we need to study moduli spaces carefully.

Current Status : Good understanding of $\Sigma(S^2)$ or point defect : X^3 - point

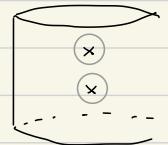
\rightsquigarrow Chiral ring = coordinate ring of Coulomb branch
of the 3d N=4 theory

Consider $\Sigma \times [0, 1] \setminus \text{pt}$ or $\Sigma \times [0, 1] \setminus (\text{small}) \text{ disk } D$



$$\mathbb{Z}(\Sigma) \otimes \mathbb{Z}(S^2) \longrightarrow \mathbb{Z}(\Sigma)$$

$\mathbb{Z}(S^2)$ has a structure of a commutative algebra,
(and $\mathbb{Z}(\Sigma)$ is its representation).



↑ we do not yet
have a rigorous definition

Coulomb branch $M_C = \text{Spec}(\mathbb{Z}(S^2) + \text{this multiplication})$.

M_C , defined in this way, is an affine algebraic variety.

Caveat: $\mathbb{Z}(S^2)$ is ∞ -dimensional. (violates an axiom of TQFT)

But it is good to have interesting M_C .

Braverman-Finkelberg-Nakajima :

$$\mathbb{Z}(S^2) = H_*(\text{moduli stack on a variant of } S^2)$$

└ modification at a point.

$D^2 \times (-\varepsilon, \varepsilon) \setminus \text{point}$



ε : small

⇒ moduli stack of holomorphic \mathbb{G} -bundles + \mathbb{N} -valued holomorphic sections ($\mathbb{C} \mathbb{M} = \mathbb{N} \otimes \mathbb{N}^*$)

monopole operators
in physics

over $D \sqcup D$

D^*

$D = \text{Spec } \mathbb{C}[[z]]$ formal disk

$D^* = \text{Spec } \mathbb{C}((z))$ formal punctured disk

\mathbb{G} = complexification of a gauge group G_C
ex.
 $= GL_n$

If we forget sections, we get a quotient stack

$$\overbrace{\text{Map}(D, G)}^{} \backslash \overbrace{\text{Map}(D^* \rightarrow G)}^{} / \overbrace{\text{Map}(D \rightarrow G)}^{} \quad \text{Map}(D^* \rightarrow G) / \text{Map}(D \rightarrow G)$$

Gr_G = affine Grassmannian

∞ -dimensional analog of
partial flag variety

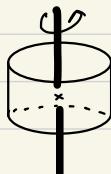
This is an important object in
geometric representation theory

Affine Grassmannian has been studied in Geometric Representation Theory.

In particular, the special case $M=0$ was studied earlier by Bezrukavnikov-Finkelberg-Mirkovic.

We succeeded to extend their result to general $M=N \oplus N^*$.

In turn, our construction (including non-commutative deformation of Coulomb branches M_C) gives new objects to be explored in Representation Theory.



add rotational symmetry

$H_*^{S^1}$ (moduli) : noncommutative
deformation of M_C

Problem. Study representations of noncomm. deform.

THANK YOU VERY MUCH !